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Multimodal Optimization of Structures with Frequency Constraints

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Optimally designed structures under eigenvalue constraints (natural frequency or buckling loads) are likely to display multiple eigenvalues. This introduces singularity of eigenvalue derivatives with respect to the design vector, which does not allow use of the Kuhn–Tucker conditions. The paper presents the optimality criteria which takes that singularity into account and, in contrast to commonly used Kuhn–Tucker conditions, they are valid for multiple eigenvalues (the form of that criteria is more general than the one derived from the Kuhn–Tucker conditions). An algorithm is presented, which applies those criteria and also uses a new method of computation of Lagrange multipliers. The algorithm allows for an arbitrary number of eigenvectors to be considered in optimality criteria, and the actual modality of the problem is determined automatically. Examples of optimization of truss structures modelled by finite element method are included.

Nomenclature

h = vector of design variables of dimension n
h_l = design variable
I = identity matrix
K = structural stiffness matrix of dimension $N \times N$
M = structural mass matrix of dimension $N \times N$
m = multiplicity of the eigenvalue
N = number of degrees of freedom of a structure
n = number of design variables
R = rotation matrix of dimension $m \times m$
r_{ij} = element of rotation matrix
V = volume of structure
X = matrix of m M-orthogonal eigenvectors of multiple eigenvalue
x_j = jth eigenvector, mode of free vibrations
β = size of optimization step
γ_{ij}, k = Lagrange's a multipliers
ΔH = linear subspace of variations of a design vector in which the multiplicity of the eigenvalue does not change
δh = element of subspace ΔH
δ_{ij} = Kronecker's delta
δx = perturbation of variable x
ε_l = residual of lth optimality condition
Λ = diagonal matrix of eigenvectors
λ_j = jth eigenvalue, square of natural frequency

Introduction

OPTIMIZATION of structures for maximum stability or maximum fundamental frequency involves solution of an eigenvalue problem. The goal of such optimization is to increase the lowest eigenvalue to its highest level for a given weight (volume) of the structure. This problem, in general terms, is equivalent to minimization of the weight of a structure subjected to buckling or frequency constraints. In this paper the approach considering maximization of eigenvalues is chosen. Optimization of natural frequencies is considered, although the presented methods are also applicable to optimization involving buckling problems.

Structural optimization leads very often to multimodal structure, because when the fundamental eigenvalue increases, the next ones usually decrease. Because of that a large number of publications have been devoted to the problem of multimodal structural optimization[1–11] in the last 15 years. All of them, except for the paper by Masur and Mroz,[2] and later works of Masur[3,4] were based on the assumption that the multiple eigenvalue is differentiable in the multidimensional design space, and the Kuhn–Tucker conditions were used to derive the optimality equations. However, those conditions are valid only when both the objective function and the constraints are Frechet differentiable; therefore, the differentiability of the eigenvalues is required both in the case of maximization of eigenvalues under the weight constraint and in the case of minimization of weight under the eigenvalue constraints. Unfortunately, that requirement is not satisfied when the eigenvalue is multiple. Although the partial derivative of multiple eigenvalue with respect to each particular design variable exists, the Frechet derivative in the multidimensional design space is not defined in general.[2,12] Therefore, the Kuhn–Tucker conditions cannot be used for derivation of the optimality conditions in the multimodal case. Apart from this, a special procedure is required to compute the partial derivatives of a multiple eigenvalue,[12,13] and even that has not been taken into account.

Masur[3] derived true optimality conditions for the problem with multiple eigenvalue constraints. However, he did not propose any solution algorithm, nor were his results used in other papers. The purpose of this paper is to present effective optimality conditions, which can be used in the optimization of structures. An algorithm is presented, which applies those conditions and also uses a new method of computation of Lagrange multipliers. The developed algorithm allows for an arbitrary number of eigenvectors to be considered in optimality criteria, and the actual modality of the problem is determined automatically.

Eigenvectors of Multiple Frequency of Free Vibrations

The squares λ_i of the natural frequencies and the corresponding modes x_j of free vibrations are the solutions of the following eigenvalue problem:

$$Kx_j = \lambda_j M x_j \qquad (j = 1, \ldots, N) \tag{1}$$

where K is the stiffness matrix and M the mass matrix. The dimension N of each eigenvector x_j is equal to the number of degrees of freedom of the system.

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If for a certain value λ_j there are m linearly independent vectors $\boldsymbol{x}_j$ satisfying Eq. (1), then the eigenvalue λ_j has multiplicity m. For simplicity let us assume that the first m eigenvalues are equal

$$\lambda_1 = \lambda_2 \ldots = \lambda_m \equiv \lambda \qquad \text{or}$$

$$\lambda_j = \lambda \qquad \text{for} \qquad j = 1, \ldots, m$$

The m linearly independent eigenvectors corresponding to a multiple eigenvalue are not defined unequivocally because any linear combination of them also satisfies Eq. (1), which means that such a combination constitutes an eigenvector too. Therefore, all of the eigenvectors form an m-dimensional subspace of the vector space $\boldsymbol{R}^N$. Any set of m linearly independent eigenvectors forms a base of that subspace and can be M orthonormalized to satisfy the following condition:

$$\boldsymbol{x}_i^\top M \boldsymbol{x}_j = \delta_{ij}, \qquad (i, j = 1, \ldots, m) \tag{2}$$

where δ_{ij} is the Kronecker's delta

$$\delta_{ij} = \begin{cases} 1 \text{ for } i = j \\ 0 \text{ for } i \neq j \end{cases}$$

Combining Eqs. (1) and (2) yields

$$\boldsymbol{x}_i^\top K \boldsymbol{x}_j = \lambda_j \delta_{ij} \qquad (i, j = 1, \ldots, m) \tag{3}$$

Equations (2) and (3) can be presented in matrix form as follows:

$$\begin{aligned} X^\top M X &= I \\ X^\top K X &= \Lambda \end{aligned} \tag{4}$$

where $X = [\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_m]$ is the matrix of eigenvectors ($N \times m$), I is the unit matrix of dimensions $m \times m$, and

$$\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_m) = \lambda I$$

The set of M-orthonormal eigenvectors X is nonunique. Using any rotation matrix R of dimensions $m \times m$, i.e., such a matrix that

$$R^\top R = R R^\top = I \tag{5}$$

the set of vectors X can be transformed into $X' = [\boldsymbol{x}'_1, \ldots, \boldsymbol{x}'_m]$ in the following way:

$$X' = XR \tag{6}$$

Matrix X' satisfies the same equations as Eq. (4), which can be proven by the following transformations:

$$\begin{aligned} X'^\top M X' &= R^\top X^\top M X R = R^\top R = I \\ X'^\top K X' &= R^\top X^\top K X R = R^\top \lambda I R = \lambda I \end{aligned} \tag{7}$$

It means that vectors $\boldsymbol{x}'_i (i = 1, \ldots, m)$, being the linear combination of the eigenvectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m$, also satisfy Eqs. (1–3), which are rewritten as

$$K \boldsymbol{x}'_i = \lambda_i M \boldsymbol{x}'_i \tag{8}$$

$$\boldsymbol{x}_i'^\top M \boldsymbol{x}'_j = \delta_{ij} \qquad (i, j = 1, \ldots, m) \tag{9}$$

$$\boldsymbol{x}_i'^\top K \boldsymbol{x}'_j = \lambda_j \delta_{ij} \tag{10}$$

Sensitivity Analysis of Multiple Eigenvalues

The matrices K and M of an optimized structure are functions of the design variables $\boldsymbol{h} = [h_1, \ldots, h_n]$, as are the eigenvalues and eigenvectors. Assuming that the design vector $\boldsymbol{h}$ is perturbed by an infinitesimally small vector $\delta \boldsymbol{h} = [\delta h_1, \ldots, \delta h_n]$, the formulas for the increments $\delta\lambda_1, \ldots, \delta\lambda_m$ of the multiple eigenvalues $\lambda_1 = \cdots = \lambda_m$ can be derived. In the following paragraphs an approach similar to Masur[3] is presented for a set of matrix equations of the vibrating system. These equations are also true for any other multiple eigenvalue problem, e.g., occurring in linear buckling analysis.

Let δM, δK, $\delta \boldsymbol{x}'_j$ $(j = 1, \ldots, m)$ denote the increments of matrix M, K, and vector $\boldsymbol{x}'_j$ due to the change $\delta \boldsymbol{h}$ of the design vector. Perturbing Eq. (8) and premultiplying by $\boldsymbol{x}_j'^\top$ results in

$$\boldsymbol{x}_j'^\top \delta K \boldsymbol{x}'_i + \boldsymbol{x}_j'^\top K \delta \boldsymbol{x}'_i = \delta\lambda_i \boldsymbol{x}_j'^\top M \boldsymbol{x}'_i + \lambda_i \boldsymbol{x}_j'^\top \delta M \boldsymbol{x}'_i + \lambda_i \boldsymbol{x}_j'^\top M \delta \boldsymbol{x}'_j \tag{11}$$

Substituting j for i in Eq. (8) and premultiplying by $\delta \boldsymbol{x}_i'^\top$ yields

$$\left(\delta \boldsymbol{x}_i'^\top\right) K \boldsymbol{x}'_j = \lambda_j \left(\delta \boldsymbol{x}_i'^\top\right) M \boldsymbol{x}'_j \qquad (i, j = 1, \ldots, m) \tag{12}$$

Making use of Eqs. (12), (9), the symmetry of the matrices K and M, and the fact that $\lambda_i = \lambda_j$, Eq. (11) is reduced to

$$\delta_{ij} \delta\lambda_i = \boldsymbol{x}_j'^\top (\delta K - \lambda_i \delta M) \boldsymbol{x}'_i \tag{13}$$

For $i \neq j$, δ_{ij} is zero, and the left-hand side of Eq. (13) is zero

$$0 = \boldsymbol{x}_j'^\top (\delta K - \lambda_i \delta M) \boldsymbol{x}'_i$$

$$(i \neq j;\ \ i, j = 1, \ldots, m) \tag{14}$$

In general, Eq. (14) is not satisfied by any set of M-orthogonal eigenvectors, only by a particular selection $\boldsymbol{x}'_i (i = 1, \ldots, m)$.

Having found a set of eigenvectors $\boldsymbol{x}'_i$ satisfying Eq. (14) the increments $\delta\lambda_i (i = 1, \ldots, m)$ can be determined from Eq. (13) after substituting i for j

$$\delta\lambda_i = \boldsymbol{x}_i'^\top (\delta K - \lambda_i \delta M) \boldsymbol{x}'_i \qquad (i = 1, \ldots, m) \tag{15}$$

The increment $\delta\lambda_i$ may be different for each $i = 1, \ldots, m$, therefore, the multiple eigenvalue may separate into distinct eigenvalues as a result of the increment $\delta \boldsymbol{h}$ of the design variables.

The set of eigenvectors X' satisfying Eq. (14) can be derived from any set of orthonormal eigenvectors X by selecting a proper rotation matrix R used in transformation (6). Equation (14) implies that the matrix

$$\delta\Lambda' = X'^\top (\delta K - \lambda \delta M) X' \tag{16a}$$

must be diagonal, and according to Eq. (15) the diagonal elements are equal to the increments $\delta\lambda_i$ of the eigenvalues: $\delta\Lambda' = \text{diag}(\delta\lambda_1, \delta\lambda_2, \ldots, \delta\lambda_m)$. As was already indicated, matrix $\delta\Lambda'$ is diagonal only for a special set X' of the eigenvectors, whereas a matrix $\delta\Lambda$ computed according to Eq. (16a) for any set X of the orthonormal eigenvectors is not diagonal in general

$$\delta\Lambda = X^\top (\delta K - \lambda \delta M) X \tag{16b}$$

However, having calculated matrix $\delta\Lambda$, a rotation matrix R can be computed which transforms $\delta\Lambda$ into the diagonal matrix $\delta\Lambda'$. Using transformation (6) and Eq. (16b), Eq. (16a) can be presented as

$$\delta\Lambda' = R^T \delta\Lambda R \tag{17}$$

Since matrix $\delta\Lambda$ is symmetric, then according to the spectral theorem there exists such an orthogonal matrix R which transforms matrix $\delta\Lambda$ into the diagonal matrix $\delta\Lambda'$ using the given formula. The columns of the rotation matrix R are the eigenvectors of matrix $\delta\Lambda$, so solving the eigenproblem (17) and using Eq. (6) the desired set of eigenvectors X' is obtained. The desired increments $\delta\lambda_i$ of the eigenvalues λ_i of the problem defined by Eq. (1) are the eigenvalues of matrix $\delta\Lambda$.

Matrix $\delta\Lambda$ depends on the increments δK and δM which are functions of $\delta \boldsymbol{h}$. Therefore, both the rotation matrix R and the special set of eigenvalues X' for which matrix $\delta\Lambda'$ is diagonal, are functions of $\delta \boldsymbol{h}$. Assuming that K and M are differentiable functions of the design variables, the increments δK and δM are linear functions of $\delta \boldsymbol{h}$ and the elements $\delta \Lambda_{ij}$ of the matrix $\delta\Lambda$ defined by Eq. (16b) can be written in the form

$$\delta \Lambda_{ij} = \sum_{\ell=1}^{n} \omega_{ij}^{\ell} \delta h_\ell \qquad (i, j = 1, \ldots, m) \tag{18}$$

where

$$\omega_{ij}^{\ell} = \boldsymbol{x}_i^{\mathrm{T}} \left(\frac{\partial K}{\partial h_\ell} - \lambda \frac{\partial M}{\partial h_\ell} \right) \boldsymbol{x}_j \tag{19}$$

For vectors $\boldsymbol{X}'$ satisfying Eq. (14), the diagonal elements $\delta\Lambda'_{ii}$ are equal to $\delta\lambda_i$ and the rest of the elements of matrix $\delta\Lambda'$ must be equal to zero:

$$\delta\lambda_i = \delta\Lambda'_{ii} = \sum_{\ell=1}^{n} \omega_{ii}^{\prime\ell} \delta h_\ell \qquad (i = 1, \ldots, m) \tag{20a}$$

$$\delta\Lambda'_{ij} = \sum_{\ell=1}^{n} \omega_{ij}^{\prime\ell} \delta h_\ell = 0 \qquad (i \neq j; \quad i, j = 1, \ldots, m) \tag{20b}$$

where

$$\omega_{ij}^{\prime\ell} = \boldsymbol{x}_i^{\prime\mathrm{T}} \left(\frac{\partial K}{\partial h_\ell} - \lambda \frac{\partial M}{\partial h_\ell} \right) \boldsymbol{x}'_j \tag{20c}$$

As can be seen from Eq. (18), for a given set of eigenvectors $\boldsymbol{X}$ the matrix $\delta\Lambda$ depends linearly on $\delta\boldsymbol{h}$. However, the increments $\delta\lambda_i$ are not linear functions of $\delta\boldsymbol{h}$ because according to Eq. (20a), $\delta\lambda_i$ depends on $\delta\boldsymbol{h}$ both directly and through the terms $\omega_{ii}^{\prime\ell}$. The relationship between $\omega_{ii}^{\prime\ell}$ and $\delta\boldsymbol{h}$ is implicit, since Eq. (20c) defining $\omega_{ii}^{\prime\ell}$ contains a set of eigenvectors $\boldsymbol{X}'$, which is specially selected for a given vector $\delta\boldsymbol{h}$ to satisfy Eq. (14). Because of that nonlinearity, the following nonequality holds in general:

$$\delta\lambda_i \neq \sum_{\ell=1}^{n} \frac{\partial \lambda_i}{\partial h_\ell} \delta h_\ell \tag{21}$$

The preceding nonequality would become an equality and the increment $\delta\lambda_i$ would be a linear function of $\delta\boldsymbol{h}$ if the eigenvalue λ was a differentiable function of the design variables in the vector space $\boldsymbol{R}^n$. But the eigenvalues are not differentiable in the Frechet sense, although the directional derivatives exist and can be computed from Eqs. (20). In particular, the partial derivatives exist and are given by

$$\frac{\partial \lambda_i}{\partial h_\ell} = \omega_{ii}^{\prime\ell} \tag{22}$$

The existence of those partial derivatives probably has been the reason for the common erroneous assumption that a repeated eigenvalue is a differentiable function in the design space which ultimately lead to the application of the Kuhn–Tucker optimality conditions. However, it can be proved that if the increments $\overline{\delta h}$ are confined to a subspace ΔH in which the multiplicity of the eigenvalue λ does not change, then the increments of the eigenvalues are linear functions of $\overline{\delta h}$. Vector $\overline{\delta h}$ belongs to ΔH if $\delta\lambda_1(\overline{\delta h}) = \cdots = \delta\lambda_m(\overline{\delta h})$, which means that the matrix $\delta\Lambda'$ is proportional to the identity matrix

$$\delta\Lambda' = \delta\lambda_1 I \tag{23}$$

Because of this simple form of matrix $\delta\Lambda'$, any rotation matrix R is a solution of the eigenvalue problem, Eq. (17), and, therefore, any set of orthonormal eigenvectors $\boldsymbol{x}_i$ can be used in Eqs. (20a) and (20c) to compute the increment $\delta\lambda_1 = \delta\lambda_i$. Because of this, the coefficients ω_{ii}^{ℓ} can be used as $\omega_{ii}^{\prime\ell}$ in Eq. (20a):

$$\delta\lambda_i = \sum_{\ell=1}^{n} \omega_{ii}^{\ell} \overline{\delta h_\ell} \tag{24}$$

Unlike $\omega_{ij}^{\prime\ell}$, the coefficients ω_{ij}^{ℓ} are computed for a selected, fixed set of eigenvectors X that does not depend on $\overline{\delta h}$. Therefore, Eq. (24) proves that $\delta\lambda_i$ is indeed a linear function of $\overline{\delta h}$. In other words, if vector $\overline{\delta h}$ belongs to the subspace $\Delta\boldsymbol{H}$, any set of M-orthogonal eigenvectors X can be used as X' in Eq. (20c).

According to Eq. (23), vector $\overline{\delta h}$ belongs to the subspace $\Delta\boldsymbol{H}$ if for any orthogonal set of eigenvectors the following conditions are met:

$$\begin{aligned} \delta\Lambda_{ij} &= 0 \qquad (i \neq j; \ i, j = 1, \ldots, m) \\ \delta\lambda_i - \delta\lambda_1 &= 0 \qquad (i = 2, \ldots, m) \end{aligned} \tag{25}$$

By using Eqs. (18) and (24), Eq. (25) can be presented in the following form:

$$\begin{aligned} \sum_{\ell=1}^{n} \omega_{ij}^{\ell} \overline{\delta h_\ell} &= 0 \qquad (i \neq j; \quad i, j = 1, \ldots, m) \\ \sum_{\ell=1}^{n} \left(\omega_{ii}^{\ell} - \omega_{11}^{\ell} \right) \overline{\delta h_\ell} &= 0 \qquad (i = 2, \ldots, m) \end{aligned} \tag{26}$$

If the conditions in Eq. (26) are met by $\overline{\delta h}$ for ω_{ij}^{ℓ} computed for one set of orthonormal eigenvectors X, they also hold for any other such set. Equation (26) forms a complete set of conditions defining elements of the subspace $\Delta\boldsymbol{H}$ in which the multiplicity of the eigenvalue does not change.

Optimality Conditions

The goal of this paper is to find a point $\boldsymbol{h}$ in the n-dimensional design space which provides the highest value of the fundamental frequency of a structure of given volume V_0. The fundamental frequency may be multiple at the optimum, which happens quite often, even if the fundamental frequency of the initial structure is single. The multiplicity m of the fundamental frequency usually increases in the process of optimization, and the actual modality of the problem can be determined when the solution approaches the optimum. The algorithm presented in the next section allows for automatic detection of the modality of the problem. That optimization problem is equivalent to minimization of the weight of the structure with multiple frequency constraints.

At the point of optimum, the following condition holds:

$$\delta\lambda_1(\delta\boldsymbol{h}) \leq 0 \tag{27}$$

for any small enough increment $\delta\boldsymbol{h}$ of the design vector ($\|\delta\boldsymbol{h}\| < \varepsilon$) that does not change the volume of the structure, i.e.,

$$\delta V(\delta\boldsymbol{h}) = \sum_{\ell=1}^{n} \frac{\partial V}{\partial h_\ell} \delta h_\ell = 0 \tag{28}$$

The optimal solution is m modal if the following additional conditions are satisfied at point $\boldsymbol{h}$ (but not necessarily in its vicinity, even in an infinitesimal region):

$$\lambda_j(\boldsymbol{h}) = \lambda_1(\boldsymbol{h}) \qquad (j = 2, \ldots, m) \tag{29}$$

In our subsequent considerations the vector $\delta\boldsymbol{h}$ will be limited to the subspace $\Delta\boldsymbol{H}$ in which the modality of the first eigenvalue is maintained, i.e.,

$$\delta\lambda_j(\delta\boldsymbol{h}) = \delta\lambda_1(\delta\boldsymbol{h}) \qquad (j = 2, \ldots, m) \tag{30}$$

in that case, as it was proved in the previous paragraph, $\delta\lambda_j$ is a linear function of $\delta\boldsymbol{h}$ and the following is true:

$$\delta\lambda_j(-\delta\boldsymbol{h}) = -\delta\lambda_j(\delta\boldsymbol{h}) \qquad (j = 2, \ldots, m) \tag{31}$$

Of course, if the vector $\delta\boldsymbol{h}$ satisfies condition (28) for keeping the volume constant, the vector $-\delta\boldsymbol{h}$ also satisfies that condition. As there are no additional constraints, admissibility of vector $\delta\boldsymbol{h}$ also implies admissibility of vector $-\delta\boldsymbol{h}$. Therefore, taking into account Eq. (31), the inequality in Eq. (27) has to be ruled out. Using

Eq. (24), the optimality conditions (27) can be presented in the form

$$\delta\lambda_1(\delta \boldsymbol{h}) = \sum_{\ell=1}^{n} \omega_{11}^{\ell}\delta h_\ell = 0 \tag{32}$$

for each $\delta \boldsymbol{h}$ satisfying both constant volume conditions (28) and the constraints (26) which define elements of the subspace $\Delta \boldsymbol{H}$. Using Lagrange multipliers k and $\gamma_{ij}(i, j = 1, \ldots, m$, except for $i = j = 1)$, the optimality condition (32), with the constraints (28) and (26) on $\delta \boldsymbol{h}$, can be written as the following condition without any constraints on $\delta \boldsymbol{h}$:

$$\sum_{\ell=1}^{n} \omega_{11}^{\ell}\delta h_\ell + \sum_{i=2}^{m}\sum_{\ell=1}^{n} \gamma_{ii}\left(\omega_{ii}^{\ell} - \omega_{11}^{\ell}\right)\delta h_\ell + \sum_{\substack{i,j=1\\ i\neq j}}^{m} \gamma_{ij}\sum_{\ell=1}^{n}\omega_{ij}^{\ell}\delta h_\ell + k\sum_{\ell=1}^{n}\frac{\partial V}{\partial h_\ell}\delta h_\ell = 0 \tag{33}$$

Introducing γ_{11} defined as

$$\gamma_{11} = 1 - \sum_{i=2}^{m}\gamma_{ii} \tag{34}$$

and taking into account the fact that Eq. (33) must hold for any set of $\delta h_\ell(\ell = 1, \ldots, n)$, the following set of n optimality conditions is derived:

$$\sum_{i,j=1}^{m} \gamma_{ij}\omega_{ij}^{\ell} + k\frac{\partial V}{\partial h_\ell} = 0 \qquad (\ell = 1, \ldots, n) \tag{35}$$

where ω_{ij}^{ℓ} is defined by Eq. (19).

Equation (35) represents general optimality conditions which take into account the fact that the multimodal frequencies are not differentiable in the design space. All other papers, except the papers by Masur,[1,2] assumed existence of those derivatives, and using the Kuhn–Tucker conditions derived the optimality conditions which do not contain factors ω_{ij}^{ℓ} for $i \neq j$ and generally are false.

It will be proved, however, that the Lagrange coefficients γ_{ij} vanish for $i \neq j$ when a special selection of eigenvectors is used for computation of the terms ω_{ij}^{ℓ}. Using the rotation (6) of the eigenvectors it can be shown that there is a rotation matrix R which transforms the matrix $[\gamma_{ij}]$ of the Lagrange coefficients into a diagonal matrix $[\gamma'_{ij}]$. As the result of this, the optimality conditions (35) will assume the following form:

$$\sum_{i=1}^{m} \gamma'_{ii}\omega_{ii}^{\prime\ell} + k\frac{\partial V}{\partial h_\ell} = 0 \qquad (\ell = 1, \ldots, n) \tag{36}$$

To prove it let us first find the relationship between the terms ω_{ij}^{ℓ} and $\omega_{ij}^{\prime\ell}$ due to the rotation (6). Because of the orthogonality of the rotation matrix R, the transformation (6) can be written in the form

$$X = X'R^{\top} \qquad \text{or} \qquad \boldsymbol{x}_i = \sum_{p=1}^{m} r_{ip}\boldsymbol{x}'_p \tag{37a}$$

where r_{ip} are the elements of matrix R (ith row and pth column). Using the notation

$$A_\ell = \frac{\partial K}{\partial h_\ell} - \lambda\frac{\partial M}{\partial h_\ell} \tag{37b}$$

and applying transformation (37b), the formula (19) can be expressed as

$$\omega_{ij}^{\ell} = \boldsymbol{x}_i^{\top}A_\ell \boldsymbol{x}^j = \sum_{p=1}^{m} r_{ip}(\boldsymbol{x}'_p)^{\top}A_\ell\sum_{q=1}^{m} r_{jq}\boldsymbol{x}'_q = \sum_{p,q=1}^{m} r_{ip}\omega_{pq}^{\prime\ell}r_{jq} \tag{37c}$$

Substituting the preceding relationship between the terms ω_{ij}^{ℓ} and $\omega_{ij}^{\prime\ell}$ into the optimality conditions (36) we obtain

$$\sum_{p,q=1}^{m} \gamma'_{pq}\omega_{pq}^{\prime\ell} + k\frac{\partial V}{\partial h_\ell} = 0 \qquad (\ell = 1, \ldots, n) \tag{37d}$$

where

$$\gamma_{pq} = \sum_{i,j=1}^{m} r_{ip}\gamma'_{ij}r_{jq} \tag{37e}$$

The relationship between γ'_{ij} and γ_{pq} can be written in a matrix form

$$[\gamma_{ij}] = R[\gamma'_{ij}]R^{\top} \qquad \text{or} \qquad [\gamma'_{ij}] = R^{\top}[\gamma_{ij}]R \tag{37f}$$

Because of the symmetry of matrix $[\gamma_{ij}]$, there is always an orthogonal matrix R which transforms it into a diagonal matrix $[\gamma'_{ij}]$ (the columns of the matrix R are the eigenvectors of matrix $[\gamma_{ij}]$). It proves that a diagonal form of the optimality conditions, Eq. (36), exists but only for a particular selection of eigenvectors X, which has to be known a priori, or some additional conditions on those eigenvectors are imposed.

As can be easily noticed from the definition (19), the terms ω_{ij}^{ℓ} are symmetric with respect to i and j, i.e., $\omega_{ij}^{\ell} = \omega_{ji}^{\ell}$. Therefore, the Lagrange coefficients γ_{ij} and γ_{ji} can be combined together in the optimality condition (35) which, after division by $-k(\partial V/\partial h_\ell)$, assumes the following form:

$$\sum_{i\geq j=1}^{m} \overline{\gamma}_{ij}e_{ij}^{\ell} - 1 = 0 \qquad (\ell = 1, \ldots, n) \tag{38}$$

where

$$\overline{\gamma}_{ii} = \frac{\gamma_{ii}}{-k}, \qquad \overline{\gamma}_{ij} = \frac{(\gamma_{ij} + \gamma_{ji})}{-k} \qquad (i \geq j)$$

$$e_{ij}^{\ell} = \omega_{ij}^{\ell}\Big/\frac{\partial V}{\partial h_\ell} \tag{39}$$

As we see, the number of the Lagrange coefficients $\gamma_{ij}(i \neq j; i, j = 1, \ldots, m)$ has been reduced by half and replaced by $\gamma_{ij}(i > j;\ i, j = 1, \ldots, m)$. This was possible because the constraints (33b) are actually the same for (i, j) and (j, i) due to the symmetry of the terms ω_{ij}^{ℓ}.

The optimality condition in the form of Eq. (38) has a certain physical interpretation. It can be proved that the term e_{ij}^{ℓ} has units of energy density. If h_ℓ describes the size of the element ℓ, then e_{ij}^{ℓ} is a function of potential and kinetic energy of that element corresponding to modes i and j. If volume V_ℓ of element ℓ is a linear function of variable h_ℓ, which happens very often in practice, then

$$\frac{\partial V_\ell}{\partial h_\ell} = \frac{V_\ell}{h_\ell} \tag{40}$$

Also, if the mass M^ℓ and stiffness K^ℓ matrices of element ℓ are linearly dependant on h_ℓ, then according to Eq. (19),

$$\omega_{ij}^{\ell} = \boldsymbol{x}_i^{\top}\left(K^\ell - \lambda_i M^\ell\right)\boldsymbol{x}_j \cdot \frac{1}{h_\ell} = \frac{E_{ij}^{\ell}}{h_\ell} \tag{41}$$

where E_{ij}^{ℓ} is the difference between the potential and kinetic energy of element ℓ. According to Eqs. (39–41),

$$e_{ij}^{\ell} = E_{ij}^{\ell}/V_\ell \tag{42}$$

The optimality condition (38) states that a certain function of energy density has to be equal to unity for all of the elements. In other words, that function must be uniform for the whole structure.

Optimization Algorithm

The solution of the presented problem can be achieved by an iterative process. One iteration step requires solution of the eigenvalue problem (1) for the current set of the design variables h_ℓ and determination of their change δh_ℓ, which drives them toward the optimum. The resizing of the design variables is based on the optimality criterion approach, in which some special features are implemented to provide convergence in the case of multiple frequencies. The multimodal optimality criterion given in the general form of Eq. (38) can be used to compute the increments of the design variables in the following way:

$$\delta h_\ell = \beta \varepsilon_\ell h_\ell \qquad (\ell = 1, \ldots, n) \tag{43a}$$

where β is the step size and ε_ℓ is the residual of the optimality Eq. (38), which is satisfied only at the optimum and is defined as

$$\varepsilon_\ell = \sum_{p \geq q = 1}^{m} \overline{\gamma}_{pq} e_{pq}^{\ell} - 1 \qquad (\ell = 1, \ldots, n) \tag{43b}$$

The preceding formula has often been used before but never with true optimality conditions for the case of multiple frequencies. Combining Eqs. (43a) and (43b) yields

$$\delta h_\ell = \beta \left(\sum_{p \geq q = 1}^{m} \overline{\gamma}_{pq} e_{pq}^{\ell} - 1 \right) h_\ell \qquad (\ell = 1, \ldots, n) \tag{43c}$$

To use Eq. (43c) the Lagrange multipliers γ_{ij} have to be computed first. It can be done by imposing some constraints on vector δh, which depends on γ_{ij} through Eq. (43c). The following three types of the constraints are proposed to determine the Lagrange coefficients: I) the condition of constant volume, Eq. (28); II) the conditions (20b) ensuring that when the first m eigenvalues are equal, formula (20a) can be used to compute the increments $\delta \lambda_i$ of those eigenvalues; and III) the conditions which facilitate the convergence of the first m eigenvalues to a common, multiple value λ_0 of the fundamental frequency,

$$\lambda_i + \delta \lambda_i = \lambda_0 \qquad (i = 1, \ldots, m). \tag{44}$$

The value λ_0 has to be determined too. When eigenvalues $\lambda_i (i = 1, \ldots, m)$ are equal to each other, the condition (44) is equivalent to Eq. (30) which says that the increment of the multiple eigenvalue should be the same, i.e., that the multiplicity of the eigenvalue should be maintained. That condition together with the condition II confines δh to the subspace $\Delta \boldsymbol{H}$ which was defined by Eq. (26).

Now, let us derive the constraints on the Lagrange multipliers from each of the three conditions.

Condition I

Using Eq. (43c), the constant volume condition (28) is expressed as

$$\delta V = \sum_{\ell=1}^{n} \frac{\partial V}{\partial h_\ell} \partial h_\ell = \beta \sum_{\ell=1}^{n} \frac{\partial V}{\partial h_\ell} \left(\sum_{p \geq q = 1}^{m} \overline{\gamma}_{pq} e_{pq}^{\ell} - 1 \right) h_\ell = 0 \tag{45a}$$

or, after division by β

$$\sum_{p \geq q = 1}^{m} a_{pq} \overline{\gamma}_{pq} = 1 \tag{45b}$$

where

$$a_{pq} = \sum_{\ell=1}^{n} e_{pq}^{\ell} h_\ell \bigg/ \sum_{\ell=1}^{n} \frac{\partial V}{\partial h_\ell} h_\ell \tag{45c}$$

Condition II

Substituting Eq. (43c) into Eq. (20b) the following constraints are obtained (after replacing $\omega_{ij}^{\prime \ell}$ with ω_{ij}^{ℓ}):

$$\sum_{p \geq q = 1}^{m} \left(\sum_{\ell=1}^{n} \omega_{ij}^{\ell} e_{pq}^{\ell} h_\ell \right) \overline{\gamma}_{pq} = \sum_{\ell=1}^{n} \omega_{ij}^{\ell} h_\ell \qquad (i, j = 1, \ldots, m; i > j) \tag{46}$$

Condition III

Because conditions II, based on Eq. (20b), are satisfied, the formula (20a) for the increments $\delta \lambda_i$ of the multiple eigenvalue is valid. Substituting Eqs. (20a) and (43c) to Eq. (44) one obtains

$$\lambda_i + \sum_{\ell=1}^{n} \omega_{ii}^{\ell} \beta \left(\sum_{p \geq q = 1}^{m} e_{pq}^{\ell} \overline{\gamma}_{pq} - 1 \right) h_\ell = \lambda_0 \qquad (i = 1, \ldots, m) \tag{47}$$

which, after extraction of the unknowns $\overline{\gamma}_{pq}$ and λ_0, can be rewritten in the following form:

$$\sum_{p \geq q = 1}^{m} \left(\sum_{\ell=1}^{m} \omega_{ii}^{\ell} e_{pq}^{\ell} \right) \overline{\gamma}_{pq} - \frac{1}{\beta} \lambda_0 = -\frac{\lambda_i}{\beta} + \sum_{\ell=1}^{n} \omega_{ii}^{\ell} h_\ell \tag{48}$$

There are m constraints of type (47), $m(m-1)/2$ constraints of type (46) and one constraint of type (45a), which all together make $m(m+1)/2+1$ equations. Those equations allow the unique determination of $m(m+1)/2$ Lagrange coefficients and the value of λ_0.

The modality m of the problem may change with subsequent iterations, while the frequencies converge. However, at each iteration step the modality can be determined automatically. Starting from a certain predicted (expected) value m, the Lagrange coefficients can be computed using the above algorithm. If all of the computed values γ_{ii} are positive, the solution is accepted and resizing is performed. If one of γ_{ii} is negative, the number m is decreased by 1 and the computation repeated until all γ_{ii} for $i = 1, \ldots, m$ are positive. The condition that γ_{ii} should be positive was proved by Masur.[4] It is also confirmed by numerical examples.

Another remark relates to retaining the volume of the structure. Condition I keeps the volume constant based on linear approximation of the changes of the volume due to the design variables. If the actual relationship is nonlinear or some additional constraints on the design variables are imposed, the total volume may change slightly after resizing. In that case all of the design variables should be rescaled by multiplying them by the same factor.

Examples

The presented algorithm has been implemented into a computer program which optimizes the cross-sectional area of truss elements to achieve the maximum value of the multiple fundamental frequency of a truss structure of constant volume. The finite element method (FEM) model as well as the modal analysis of free vibrations is performed using ANSYS, Engineering Analysis System developed by Swanson Analysis Systems Inc. The results from ANSYS are used by the optimization program.

The program has been used for optimization of 36 element truss presented in Fig. 1. The initial cross section of all of the elements

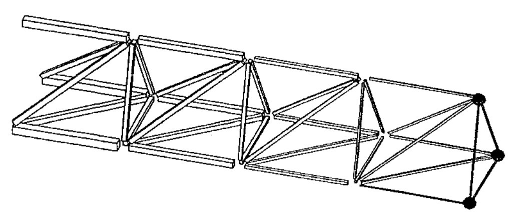

Fig. 1 After optimization, the 36-element truss, fixed at one end, 3 masses are attached at nodes at the other end.

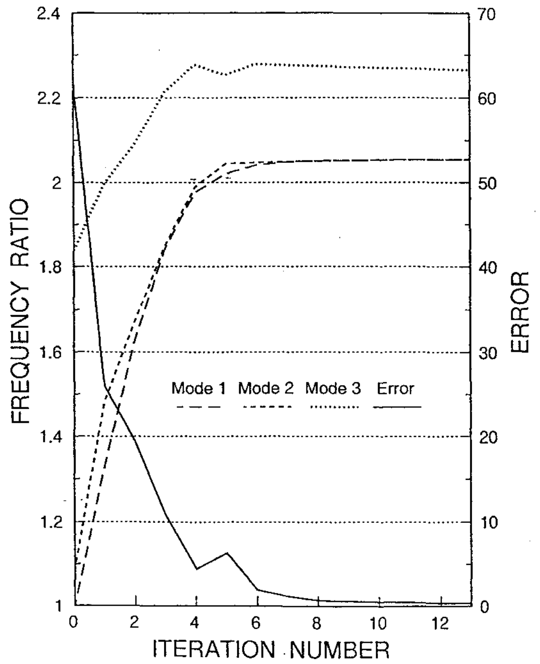

Fig. 2 History of optimization using complete optimality conditions.

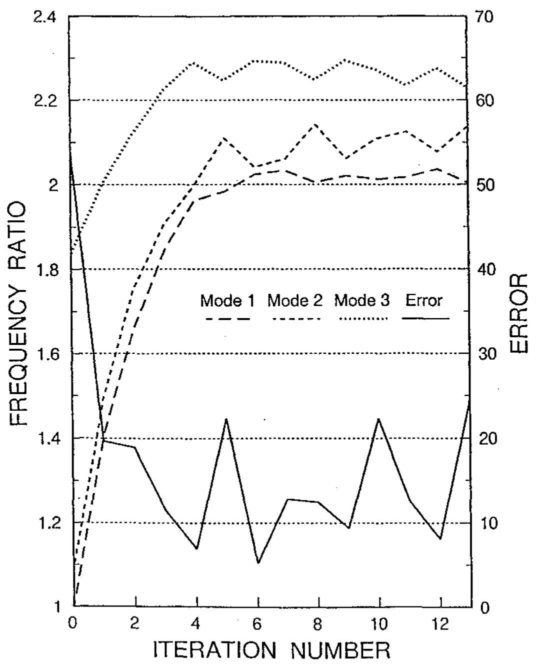

Fig. 3 History of optimization using incomplete optimality conditions.

was 2 × 2 cm. All of the longitudinal elements are 0.5 m long, but the elements forming a triangle are of different lengths: 0.4, 0.45, and 0.5 m. There are three masses, 1 kg each at one end of the structure, and three nodes on the other end which are fixed. The Young's modulus is 2.1×10^6 MPa, Poisson's ratio is 0.3, and the density is 7000 kg/m^3. The lowest frequencies of the initial (uniform) structure were: $\omega_1^0 = 55.45$ Hz, $\omega_2^0 = 60.35$ Hz $= 1.09\omega_1^0$, and $\omega_3^0 = 101.9$ Hz $= 1.84\omega_1^0$. In the result of the optimization, the two lowest frequencies became equal to each other, and they reached the value $\omega_1 = \omega_2 = 113.91$ Hz $= 2.05\omega_1^0$, which means that the fundamental frequency became bimodal and was over two times higher than the fundamental frequency of the initial structure. The third frequency became close to the bimodal fundamental frequency and reached the value $\omega_3 = 125.63$ Hz $= 2.26\omega_1^0$. Figure 2 shows the results of the optimization for each iteration step, at which only one modal analysis is performed. Figure 3 displays the ratio of three lowest frequencies to the initial fundamental frequency. It also shows

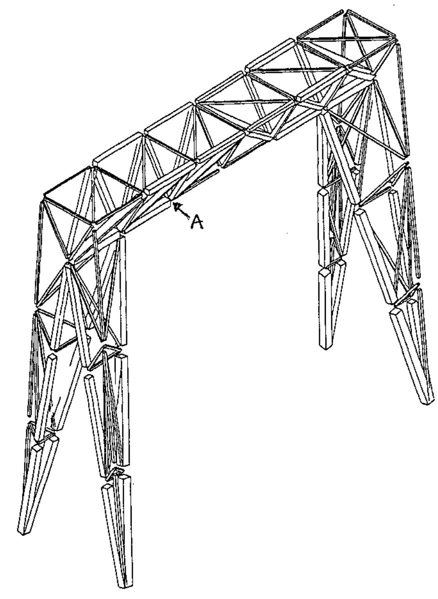

Fig. 4 Optimized truss of a crane, at point A the mass of 500 lbs was attached.

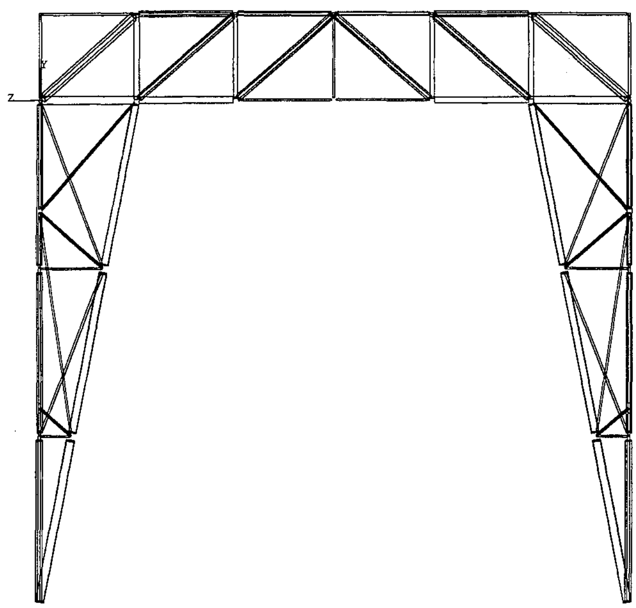

Fig. 5 Front view of the truss.

the error, which represents the maximum residual ε_ℓ of optimality conditions for all of the elements. As can be seen, the optimum was practically achieved in six iterations.

The optimization was also attempted using the assumption that the Lagrange coefficients γ_{ij} are equal to zero for all $i \neq j$. As was pointed out earlier, these generally incorrect optimality conditions were used in all the algorithms published to date. It is seen from the optimization history shown in Fig. 3 that this algorithm encounters problems when the natural frequencies are close to each other and the bimodal frequency does not fully converge.

As the second example, the 141-element truss presented in Figs. 4 and 5 was optimized. The main dimensions of the truss were: length 600 in. and height 596 in. The initial cross section of the elements was 2.25 in.^{2}. The minimal cross section was assumed equal to 0.45 in.^{2}, the density was 0.23 lbs/in.^{3}. The mass of 500 lbs was hung at the point A at the distance one third of the length of the span. The initial natural frequencies of the structure were $\overline{\omega}_0 = 0.39$ Hz, $\overline{\omega}_1 = 0.69$ Hz, and $\overline{\omega_2} = 0.81$ Hz.

The optimized structure has the natural frequencies $\omega_0 = \omega_1 = 0.58$ Hz $= 1.49\overline{\omega}_0$, $\omega_2 = 1.57\overline{\omega}_0$, and 62 elements reached the minimal cross section. The history of iterations is presented in Fig. 6. Figure 7 displays the optimization history when the incorrect

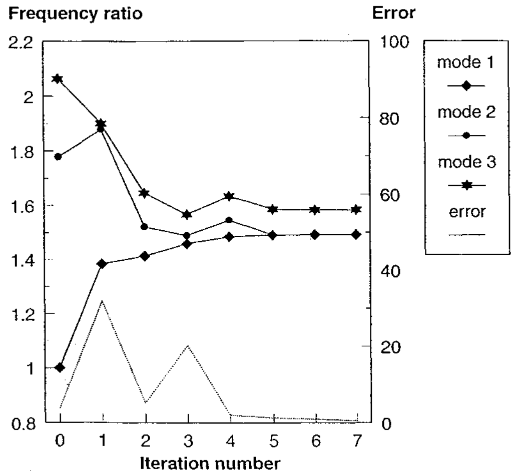

Fig. 6 History of optimization using complete optimality conditions.

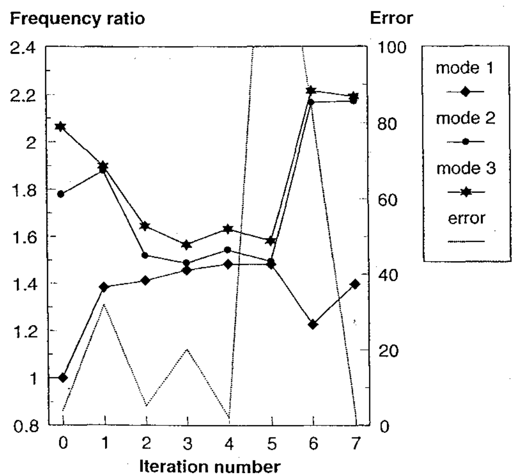

Fig. 7 History of optimization using incomplete optimality conditions.

optimality condition was used. It can be seen again, that using incomplete optimality conditions causes problems with convergence when natural frequencies become close to each other, whereas the application of the correct optimality conditions eliminates that problem.

Conclusions

The paper presents the optimality conditions that should be used while optimizing structures under multiple eigenvalue constraints. Also, the optimization algorithm has been presented, which is effective and provides fast convergence for the cases when natural frequencies become close to each other.

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